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ON INITIAL, BOUNDARY CONDITIONS AND VISCOSITY COEFFICIENT CONTROL FOR BURGERS' EQUATION

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SUMMARY

In order to use the optimal control techniques in models of geophysical flow circulation, an application to a 1D advection–diffusion equation, the so-called Burgers' equation, is described. The aim of optimal control is to find the best parameters of the model which ensure the closest simulation to the observed values. In a more general case, the continuous problem and the corresponding discrete form are formulated. Three kinds of simulation are realized to validate the method. Optimal control processes by initial and boundary conditions require an implicit discretization scheme on the first time step and a decentered one for the non-linear advection term on boundaries. The robustness of the method is tested with a noised dataset and random values of the initial controls. The optimization process of the viscosity coefficient as a time- and space-dependent variable is more difficult. A numerical study of the model sensitivity is carried out. Finally, the numerical application of the simultaneous control by the initial conditions, the boundary conditions and the viscosity coefficient allows a possible influence between controls to be taken into account. These numerical experiments give methodological rules for applications to more complex situations. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: optimal control; Burgers' equation; data assimilation; adjoint model

1. INTRODUCTION

In order to model the coastal oceanic circulation, special degenerated forms of the Navier–Stokes equations are often considered, usually called the Shallow Water equations (see Leendertse and Liu [1] or Nihoul and Jamart [2] for examples). For a 1D evolution problem, these equations take the form of a non-linear advection–diffusion model given by the viscous Burgers' equation [3], on which the present study has been focused. These equations are controlled by functional parameters that are to be fitted.

Since the pioneering work of Sasaki [4], the data assimilation method has been used in meteorology by Marchuk [5] and Le Dimet and Talagrand [6], and in oceanography by Bennet and McIntosch [7] and Provost and Salmon [8]. Ideas and mathematical concepts of optimal control theory were formalized by Lions [9] 30 years ago and have since received attention for applications in oceanography (e.g. Begis and Crepon [10] or Devenon [11]). The aim of optimal control is to find the best parameters of the model to simulate the closest computed values to the observed ones. This variational method involves a minimization of a cost function which

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is the norm of the difference between the computed and observed values. An algorithm is obtained via the so-called adjoint equations and this minimization for the construction of the cost function gradient with respect to the parameters. Once the gradient is determined, the minimization can be performed using any numerical optimization algorithm.

Some authors have already dealt with this kind of problem. Dean and Gubernatis [12] introduced a pointwise control. Lellouche *et al.* [13] focused their study on the control by boundary conditions. This approach is sufficient if it is considered that in the more common cases, the control by boundary conditions is more influential than the control by initial conditions and by the viscosity coefficient. Nevertheless, in order to use optimal control in more complex models, the three types of controls are studied for this simple equation.

In the first part, the optimal control problem will be developed for the general case of control by initial and boundary conditions and by the viscosity coefficient. The suggested method will then be validated by three phases of numerical experiments:

– the control by the initial and boundary conditions,
– the specific control by the viscosity coefficient as a distributed variable,
– the simultaneous control by the three parameters.

Perspectives and conclusions will end our reflection.

2. THE SIMULTANEOUS OPTIMAL CONTROL PROBLEM

2.1. The continuous problem

According to the previous section, the chosen numerical model is the viscous Burgers' equation in a one space dimension. If we interpret the scalar function $y(x, t)$ as modeling the velocity at a point x and a time t, then the governing equation assorted with suitable boundary and initial conditions is given by

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{1}{2}\dfrac{\partial (y^2)}{\partial x}-\dfrac{\partial}{\partial x}\left(v\dfrac{\partial y}{\partial x}\right)=0 & \text{in } Q \\ y(x,0)=u(x) & \text{for } x\in(0,L) \\ y(0,t)=Y_1(t) & \text{for } t\in(0,T) \\ y(L,t)=Y_2(t) & \text{for } t\in(0,T) \end{cases} \tag{1}$$

where $Q=(0, L)\times(0, T)$. The viscosity coefficient $v(x, t)$ can vary in space and time. The problem is to control this system in order to produce a desired state, denoted $\hat{y}$, concerning the modeled quantity y. The control procedure consists of finding the optimal control, $\Psi_{\text{opt}}=(u_{\text{opt}}, Y_{1\text{opt}}, Y_{2\text{opt}}, v_{\text{opt}})$ (and the corresponding optimal solution y_{opt}) which minimizes a cost criterion measuring the Euclidean norm of the difference between y and $\hat{y}$. In the general case,

$$y=y(\Psi)=y(u, Y_1, Y_2, v), \tag{2}$$

with $u\in L^2(0, L)$, Y_1, $Y_2\in L^2(0, T)$ and $v\in L^2(Q)$.

Considering the cost function,

$$J(\Psi)=\frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)}=\frac{1}{2LT}\int_0^T\int_0^L (y-\hat{y})^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3}$$

the problem is to find

$$\min\{J(\Psi):\ \Psi\in\xi_{\mathrm{ac}}\}, \tag{4}$$

with $\xi_{\mathrm{ac}} = L^2(0, L)\times L^2(0, T)\times L^2(0, T)\times L^2(Q)$, the set of admissible controls.

Let us introduce the directional derivative of J at $\Psi\in\xi_{\mathrm{ac}}$ along $\phi\in\xi_{\mathrm{ac}}$, given by

$$J'(\Psi, \phi) = \lim_{\delta\to 0}\frac{J(\Psi+\delta\phi)-J(\Psi)}{\delta}. \tag{5}$$

Assuming $J'(\Psi, \phi)$ exists for all ϕ and that $J'(\Psi, .\ )$ is linear and continuous in ϕ, the gradient $\nabla J(\Psi)$ will be defined as the linear form, satisfying

$$J'(\Psi, \phi) = \langle \nabla J(\Psi), \phi\rangle \quad \forall\phi\in\xi_{\mathrm{ac}}, \tag{6}$$

where $\langle .,. \rangle$ is the dual product which is also, in this case, the inner product in ξ_{ac}.

To solve Equation (4), it is not possible to resolve $J'(\Psi, \phi) = 0$ directly. We are led to identify $\nabla J(\Psi)$ from $J'(\Psi, \phi)$. From Equations (3) and (5)

$$J'(\Psi, \phi) = \langle y-\hat{y}, y'\rangle = \frac{1}{LT}\int_0^T\int_0^L (y-\hat{y})y'(\Psi, \phi)\,\mathrm{d}x\,\mathrm{d}t, \tag{7}$$

where

$$y'(\Psi, \phi) = \lim_{\delta\to 0}\frac{y(\Psi+\delta\phi)-y(\Psi)}{\delta}. \tag{8}$$

We now have to find conditions verified by y'. Let $\Psi = (u, Y_1, Y_2, v)$ and $\phi = (v, \varphi_1, \varphi_2, \alpha)$ belong to the space ξ_{ac}, putting down the two systems fulfilled by $y(\Psi)$ and $y(\Psi+\delta\phi)$

$$\begin{cases} \dfrac{\partial y(\Psi)}{\partial t}+\dfrac{1}{2}\dfrac{\partial(y^2(\Psi))}{\partial x}-\dfrac{\partial}{\partial x}\left(v(\Psi)\dfrac{\partial y(\Psi)}{\partial x}\right)=0 \quad \text{in } Q \\ y(x, 0)(\Psi) = u(x) \quad \text{for } x\in(0, L) \\ y(0, t)(\Psi) = Y_1(t) \quad \text{for } x\in(0, T) \\ y(L, t)(\Psi) = Y_2(t) \quad \text{for } x\in(0, T), \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\partial y(\Psi+\delta\phi)}{\partial t}+\dfrac{1}{2}\dfrac{\partial(y^2(\Psi+\delta\phi))}{\partial x}-\dfrac{\partial}{\partial x}\left((v(\Psi)+\delta\alpha)\dfrac{\partial y(\Psi+\delta\phi)}{\partial x}\right)=0 \quad \text{in } Q \\ y(x, 0)(\Psi+\delta\phi) = u(x)+\delta v(x) \quad \text{for } x\in(0, L) \\ y(0, t)(\Psi+\delta\phi) = Y_1(t)+\delta\varphi_1(t) \quad \text{for } t\in(0, T) \\ y(L, t)(\Psi+\delta\phi) = Y_2(t)+\delta\varphi_2(t) \quad \text{for } t\in(0, T). \end{cases} \tag{10}$$

By subtraction of Equation (9) from Equation (10) and division by δ which tends to zero, the expression verified by y', called the linearized Burgers' tangent model, $\forall\phi = (v, \varphi_1, \varphi_2, \alpha)\in\xi_{\mathrm{ac}}$ is given by

$$\begin{cases} \dfrac{\partial y'(\Psi,\phi)}{\partial t}+\dfrac{\partial (y\cdot y'(\Psi,\phi))}{\partial x}-\dfrac{\partial}{\partial x}\left(v\dfrac{\partial y'(\Psi,\phi)}{\partial x}\right)-\dfrac{\partial}{\partial x}\left(\alpha\dfrac{\partial y}{\partial x}\right)=0 \quad \text{in } Q \\ y'(\Psi,\phi)(x,0)=v(x) \quad \text{for } x\in(0,L) \\ y'(\Psi,\phi)(0,t)=\varphi_1(t) \quad \text{for } t\in(0,T) \\ y'(\Psi,\phi)(L,t)=\varphi_2(t) \quad \text{for } t\in(0,T). \end{cases} \tag{11}$$

As in Le Dimet and Talagrand [6] or Lellouche *et al.* [13], the aim is to use the conditions in Equation (11) verified by y' to evaluate Equation (7). The idea is to introduce a function, $p(x, t)$, in order to substitute ϕ for y' in Equation (7). By using Equation (11), $\nabla J(\Psi)$ is then identified as in Equation (6). The new variable $p(x, t)$, called the adjoint variable, verifies an expression of the same degree of complexity of those verified by y',

$$\begin{cases} \dfrac{\partial p}{\partial t}+y\dfrac{\partial p}{\partial x}+\dfrac{\partial}{\partial x}\left(v\dfrac{\partial p}{\partial x}\right)=y-\hat{y} \quad \text{in } Q \\ p(x,T)=0 \qquad \text{for } x\in(0,L) \\ p(0,t)=p(L,t)=0 \qquad \text{for } t\in(0,T). \end{cases} \tag{12}$$

This model, called the adjoint model, allows the calculation of J'.

$$\forall\phi=(v,\varphi_1,\varphi_2,\alpha)\in\xi_{\text{ac}},$$

$$J'(\Psi,\phi)=\frac{1}{LT}\int_0^T\int_0^L\left(\frac{\partial p}{\partial t}+y\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(v\frac{\partial p}{\partial x}\right)\right)y'(\Psi,\phi)\,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

An integration by part of Equation (13) gives an expression of J' including the conditions of (11)

$$\begin{aligned} &J'(\Psi,\phi) \\ &=\frac{1}{LT}\left[\int_0^T\int_0^L p\left(-\frac{\partial y'}{\partial t}-\frac{\partial yy'}{\partial x}+\frac{\partial}{\partial x}\left(v\frac{\partial y'}{\partial x}\right)+\frac{\partial}{\partial x}\left(\alpha\frac{\partial y}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}t+\int_0^L[y'p]_0^T\,\mathrm{d}x\right. \\ &\quad+\int_0^T[yy'p]_0^L\,\mathrm{d}t-\int_0^T\left[v\frac{\partial y'}{\partial x}p\right]_0^L\mathrm{d}t+\int_0^T\left[v\frac{\partial p}{\partial x}y'\right]_0^L\mathrm{d}t-\int_0^T\left[\alpha\frac{\partial y}{\partial x}p\right]_0^L\mathrm{d}t \\ &\quad\left.+\int_0^T\int_0^L\alpha\frac{\partial y}{\partial x}\frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}t\right], \end{aligned} \tag{14}$$

$$\begin{aligned} J'(\Psi,\phi)=&-\frac{1}{T}\left(\frac{1}{L}\int_0^L v(x)p(x,0)\,\mathrm{d}x\right)-\frac{1}{L}\left(\frac{1}{T}\int_0^T\varphi_1(t)v(0,t)\frac{\partial p(0,t)}{\partial x}\,\mathrm{d}t\right) \\ &+\frac{1}{L}\left(\frac{1}{T}\int_0^T\varphi_2(t)v(L,t)\frac{\partial p(L,t)}{\partial x}\,\mathrm{d}t\right) \\ &+\frac{1}{LT}\int_0^T\int_0^L\alpha(x,t)\frac{\partial y(x,t)}{\partial x}\frac{\partial p(x,t)}{\partial x}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{15}$$

Then the gradient's components are completely determined

$$\nabla J(\Psi) = \nabla J(u,\ Y_1,\ Y_2,\ v) = \begin{pmatrix} -\dfrac{1}{T}\,(p(x,0))_{x\in(0,L)} \\ -\dfrac{1}{L}\left(v(0,t)\dfrac{\partial p(0,t)}{\partial x}\right)_{t\in(0,T)} \\ \dfrac{1}{L}\left(v(L,t)\dfrac{\partial p(L,t)}{\partial x}\right)_{t\in(0,T)} \\ \left(\dfrac{\partial y(x,t)}{\partial x}\dfrac{\partial p(x,t)}{\partial x}\right)_{(x,t)\in Q} \end{pmatrix}. \tag{16}$$

Owing to the successive resolutions of both the direct and adjoint models, we can identify the gradient's components and build an iterative resolution method to find the optimal controls and minimize the cost function. Figure 1 summarizes this method. The minimization process of J is computed by the quasi-Newton variable storage (or limited memory) method, whose computer code was given by Gilbert and Lemarechal [14].

2.2. The discrete problem

In Section 2.1, the gradient of the cost function for a continuous problem in time and space has been expressed. In practice, it is a discrete numerical model that is only an approximation of the continuous equation.

Let the time interval $(0, T)$ (respectively, the space interval $(0, L)$) be divided into N subintervals (respectively, into $I+1$ subintervals), each of length $\Delta t = T/N$ (respectively, $\Delta x = L/(I+1)$). The discrete version of Equations (3) and (4) is then to find

$$\min\{\tilde{J}(\Psi):\ \Psi\in R^{I+2N+N\times(I+2)}\}, \tag{17}$$

with

$$\tilde{J}(\Psi) = \frac{1}{2NI}\sum_{i=1}^{I}\sum_{n=1}^{N}|y_i^n - \hat{y}_i^n|^2, \tag{18}$$

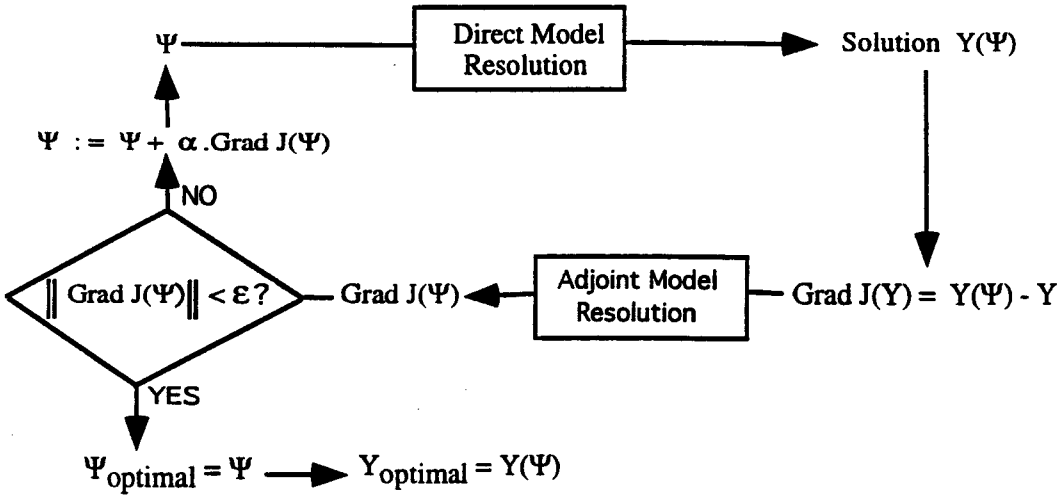

Figure 1. Sketch of the solving algorithm. J is minimized until $\|\nabla J(\Psi)\|$ is less than a threshold value ϵ.

where y_i^n (respectively, $\hat{y}_i^n$) is the approximation of $y(i\Delta x, n\Delta t)$ (respectively, the observed value y at point $(i\Delta x, n\Delta t)$).

The problem now is to choose the 'good' discretization of the two models (Equations (1) and (12)). The direct model (1) is not linear but the linearized tangent model (11) and the adjoint model (12) are. So, both can be set as discrete linear systems governed by matrices. The adjointness property between Equations (11) and (12) means that their two matrices must transpose each other. By transposing the linearized tangent matrix, the expressions of the discrete adjoint model and the discretized gradient's components can be computed. On an other hand, a 'good' discretization of the continuous problem must give the same expressions as those given by the preceeding matrix approach. Lellouche *et al.* [13] showed that for this requirement, a decentered discretization of the advection term on the boundaries is necessary. For the same reasons, to avoid unwanted terms in the gradient related to initial conditions, an implicit time discretization on the first time step must be used. A specific time discretization for the following steps is not needed. Nevertheless, an implicit scheme for every time step is chosen. This also guarantees the stability criteria. Respecting these space and time rules, a discretized direct model is given by

- Initial conditions: $y_i^0 = u_i, \quad 1 \le i \le I.$
- Boundary conditions: $y_0^n = Y_1^n$ and $y_{I+1}^n = Y_2^n, \quad 1 \le n \le N.$
- Discretization scheme, for $n = 1, \ldots, N$:

$i = 1$:

$$\frac{1}{\Delta t}(y_i^n - y_i^{n-1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(y_{i+1}^n - y_i^n) - (v_{i-1}^n + v_i^n)(y_i^n - y_{i-1}^n)) + \frac{1}{2\Delta x}(y_{i+1}^n y_{i+1}^n - y_i^n y_i^n) = 0.$$

$2 \le i \le I-1$:

$$\frac{1}{\Delta t}(y_i^n - y_i^{n-1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(y_{i+1}^n - y_i^n) - (v_{i-1}^n + v_i^n)(y_i^n - y_{i-1}^n)) + \frac{1}{4\Delta x}(y_{i+1}^n y_{i+1}^n - y_{i-1}^n y_{i-1}^n) = 0.$$

$i = I$:

$$\frac{1}{\Delta t}(y_i^n - y_i^{n-1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(y_{i+1}^n - y_i^n) - (v_{i-1}^n + v_i^n)(y_i^n - y_{i-1}^n)) + \frac{1}{2\Delta x}(y_i^n y_i^n - y_{i-1}^n y_{i-1}^n) = 0.$$

A general discretization of p_i^n, the approximation for $p(i\Delta x, n\Delta t)$ can be established for $I \ge 5$ as follows

- Final conditions: $p_i^{N+1} = 0, \quad 1 \le i \le I.$
- Boundary conditions: $p_0^n = p_{I+1}^n = 0, \quad 1 \le n \le N.$
- Discretization scheme, for $n = N, \ldots, 1$:

$i = 1$:

$$\frac{1}{\Delta t}(p_i^n - p_i^{n+1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(p_{i+1}^n - p_i^n) - (v_{i-1}^n + v_i^n)(p_i^n - p_{i-1}^n)) + \frac{1}{2\Delta x} y_i^n (2p_i^n + p_{i+1}^n) = \hat{y}_i^n - y_i^n.$$

$i = 2$:

$$\frac{1}{\Delta t}(p_i^n - p_i^{n+1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(p_{i+1}^n - p_i^n) - (v_{i-1}^n + v_i^n)(p_i^n - p_{i-1}^n)) + \frac{1}{2\Delta x} y_i^n (2p_{i-1}^n - p_{i+1}^n) = \hat{y}_i^n - y_i^n.$$

$3 \le i \le I - 2$:

$$\frac{1}{\Delta t}(p_i^n - p_i^{n+1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(p_{i+1}^n - p_i^n) - (v_{i-1}^n + v_i^n)(p_i^n - p_{i-1}^n)) + \frac{1}{2\Delta x} y_i^n (p_{i-1}^n - p_{i+1}^n) = \hat{y}_i^n - y_i^n.$$

$i = I - 1$:

$$\frac{1}{\Delta t}(p_i^n - p_i^{n+1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(p_{i+1}^n - p_i^n) - (v_{i-1}^n + v_i^n)(p_i^n - p_{i-1}^n)) + \frac{1}{2\Delta x} y_i^n (p_{i-1}^n - 2p_{i+1}^n) = \hat{y}_i^n - y_i^n.$$

$i = I$:

$$\frac{1}{\Delta t}(p_i^n - p_i^{n+1}) - \frac{1}{2\Delta x^2}((v_i^n + v_{i+1}^n)(p_{i+1}^n - p_i^n) - (v_{i-1}^n + v_i^n)(p_i^n - p_{i-1}^n)) + \frac{1}{2\Delta x} y_i^n (p_{i-1}^n + 2p_i^n) = \hat{y}_i^n - y_i^n.$$

This discretization leads to the following discretized gradient

$$\nabla \tilde{J}(\Psi) = \begin{pmatrix} -\frac{1}{\Delta t}(p_i^0)_{1 \le i \le I} \\ -\frac{1}{2\Delta x^2}((v_0^n + v_1^n)p_1^n)_{1 \le n \le N} \\ \frac{1}{2\Delta x^2}((v_I^n + v_{I+1}^n)p_I^n)_{1 \le n \le N} \\ \frac{1}{2\Delta x^2}((y_{i+1}^n - y_i^n)(p_{i+1}^n - p_i^n) + (y_i^n - y_{i-1}^n)(p_i^n - p_{i-1}^n))_{1 \le n \le N}^{1 \le i \le I} \end{pmatrix}.$$

3. NUMERICAL EXPERIMENTS

To verify the efficiency of the optimal control method described in the previous section, three experiment phases have been realized.

Controls by initial and boundary conditions are very similar and two experiments of simultaneous control were carried out. Control by a distributed viscosity coefficient presents some difficulties that are pointed out by several numerical studies on the relative model sensibility. Lastly, the effectiveness of the method is validated by an experiment simultaneously implementing the three kinds of controls.

3.1. On boundary and initial conditions

Two numerical experiments on control by initial and boundary conditions have been realized with the following coefficient values

$$v = 1 \text{ m}^2 \text{ s}^{-1}, \quad L = 10 \text{ m}, \quad T = 1 \text{ s}, \quad N = 50, \quad I = 20.$$

The implicit scheme of the discretized equations tends to solve, at each time step, a matrix system $A_n(Y_n) \cdot Y_n = B_n$, where A_n is a tridiagonal matrix which depends on the vector $Y_n = (y_i^n)_{i=1,\ldots,n}$, and B_n is a vector dependent on the previous time step vector Y_{n-1} and on the boundary conditions y_0^n and y_{I+1}^n. A classical iterative method of double swap, the so-called Thomas method, is used to solve this kind of system.

In order to produce the data set to be assimilated by the optimal control method, the direct model (1) is used to compute a solution. The controls used to simulate the data are arbitrarily chosen and called the 'true' controls. Here, $\hat{y}(x, t)$ is simulated with the 'true' controls

$$y(x, 0) = u(x) = 1 - \frac{2x}{L} \quad \text{for } x \in (0, L),$$

$$y(0, t) = \Psi_1(t) = 1 - \frac{2t}{T} \quad \text{for } t \in (0, T),$$

$$y(L, t) = \Psi_2(t) = \frac{2t}{T} - 1 \quad \text{for } t \in (0, T).$$

With the knowledge of $\hat{y}$, the aim of the optimal control method is now to find the optimal controls producing the optimal simulation.

3.1.1. Experiment 1. In this experiment, all the data at each grid point are used. To begin, the more unfavourable situation is used. Initial and boundary conditions are unknown and take random values (Gaussian noise). The more iterations are performed, the closer the controls' values and the 'true' controls are. At the end, the conditions that permitted the simulation of the data set are exactly recovered (see Figures 2 and 3). The results concerning the right boundary conditions are very similar to the left ones; so they are not presented. Dataset and optimal simulation are no longer discernible. As shown in Figure 4, the cost function, J, decreases with iterations until it is an infinitely small quantity.

The cost function is computed by the sum of differences between very close terms. So, when these differences are no longer significant for the computer (here real*4, i.e. 10e − 39), the sum, and hence the cost function, does not evolve any more and the method stops. For the numerical capabilities of the computer, the best simulation (the nearest one to the dataset) is effectively reached.

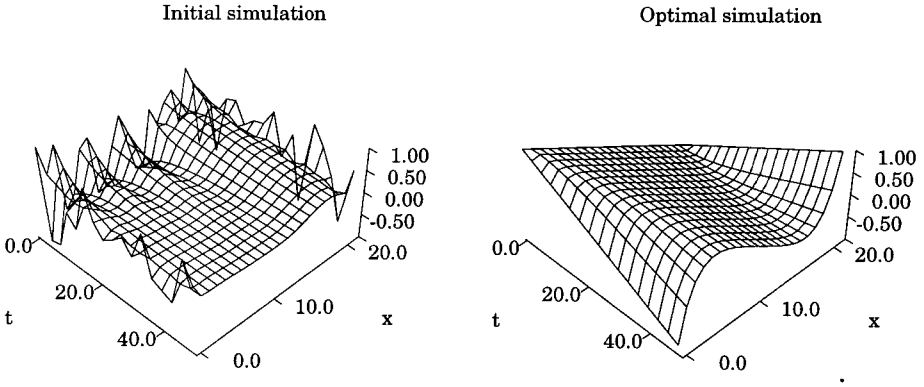

Figure 2. Initial simulation and optimal simulation which exactly correspond to the dataset simulation.

3.1.2. Experiment 2. In this second example, a more realistically noised dataset (because of geophysical, electronic noise, etc.), obtained by a noised solution of the direct model (1) introducing a Gaussian error component is considered. The dataset becomes more chaotic (see Figure 5).

The optimization process gives controls close to the desired controls. These optimal coefficients permit the simulation of the velocity field that is the closest to the noised one. Figure 6 shows the control found for initial conditions.

These two experiments show the robustness of the method. Following the same discretization rules in space and time that we have pointed out, the optimal control method may be used in more complex situations like 3D Shallow Water Navier–Stokes equations occurring in ocean circulation models.

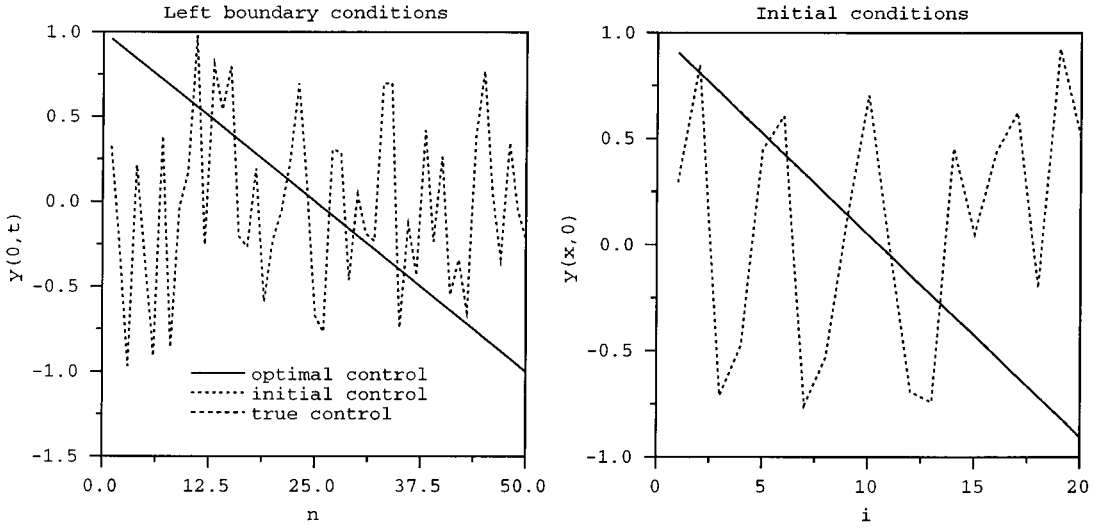

Figure 3. Left boundary and initial controls.

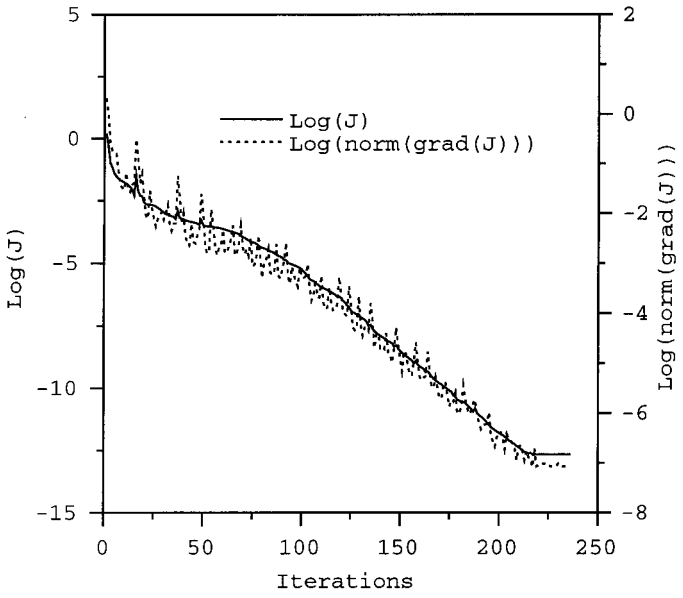

Figure 4. Variations of the scaled cost logarithm function $\log_{10} J$ and of the scaled function $\log_{10}(\|\nabla J\|)$ with the number of iterations.

3.2. On viscosity coefficient

In this part, the optimization of the viscosity coefficient is focused on. It leads to the study of the influence on the direct model solution, where initial and boundary conditions are well-known.

In order to simulate the dataset, taking again the situation of Section 3.1 for initial and boundary conditions, for a pure computation exercise an arbitrary variation of the viscosity coefficient is introduced

$$v(x, t) = 20 \frac{x}{L} \frac{t}{T} + 0.1 \quad \text{for } (x, t) \in Q.$$

Though this variation has no physical sense, the values of v range from 0.1 to 20.1 $m^2\ s^{-1}$. This leads to a large range of Reynolds number, between 0.1 and 100. It also ensures the

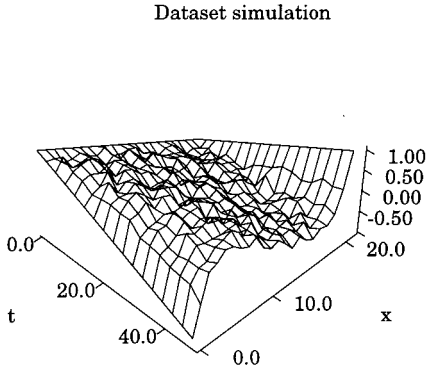

Figure 5. Noised dataset.

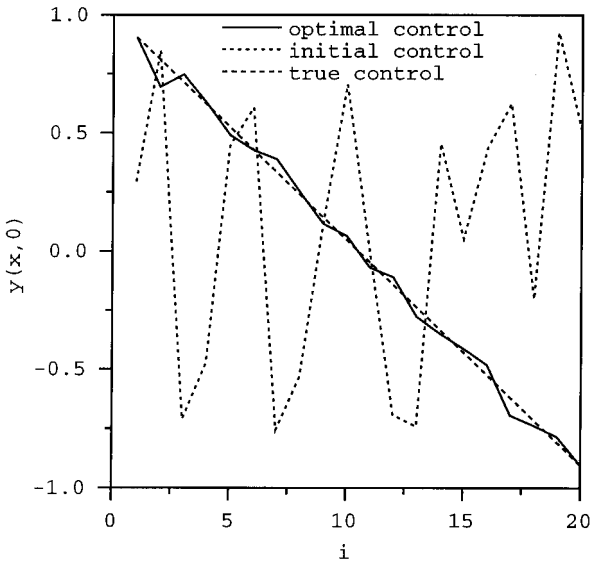

Figure 6. Initial conditions.

stability of the model. The optimal control process allows the exact viscosity coefficient from which the exact solution has been computed to be found (see Figure 7).

Nevertheless, as shown in Figure 8, the decrease of the cost function is very slow and the number of iterations is rather high with regard to that obtained when boundary and initial conditions were optimized. So, the quest for the optimal values of v is rather difficult. One can also wonder whether the coefficient values have a real influence on the model solution.

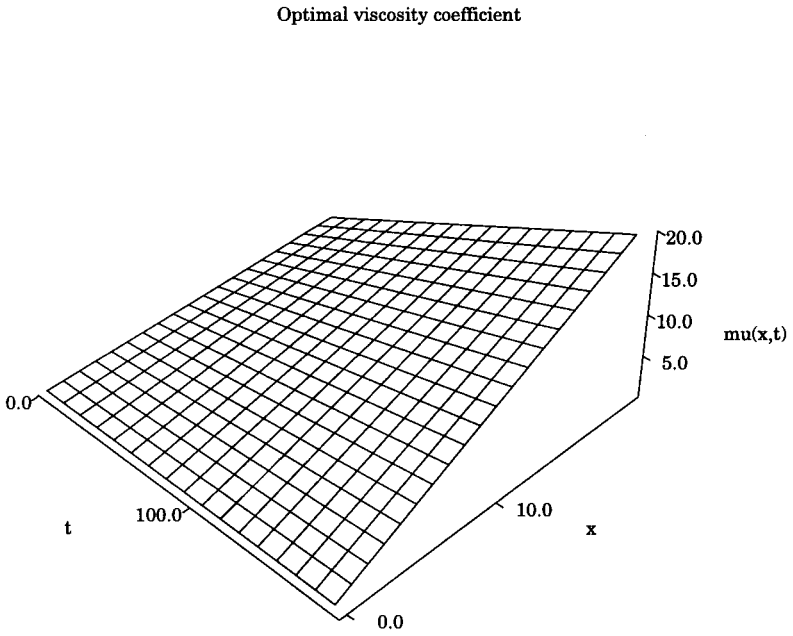

Figure 7. Optimal viscosity coefficient in the case of well-known initial and boundary conditions.

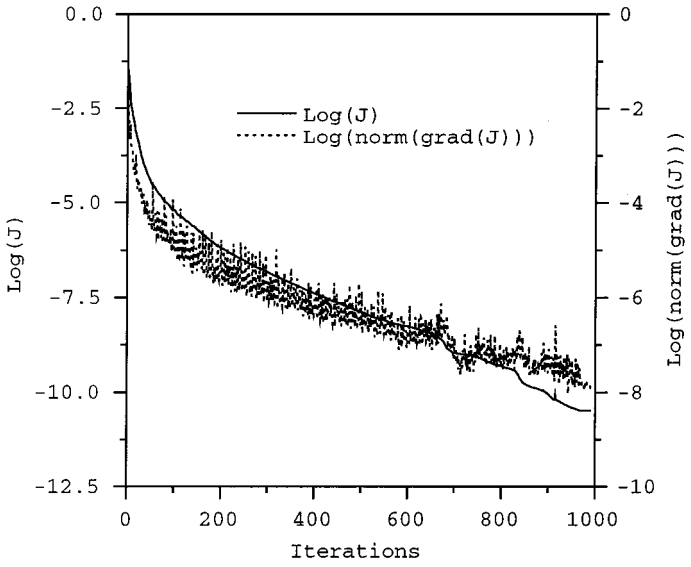

Figure 8. Variations of the scaled cost logarithm function $\log_{10} J$ and of the scaled function $\log_{10}(\|\nabla J\|)$ with $v(x, t)$ as control.

In trying to answer this question, another experiment has been realized, with v varying exclusively with space for a better visualization

$$v(x) = 20\frac{x}{L} + 0.1 \quad \text{for } x \in (0, L).$$

In the previous simulation, the optimization process was stopped by the inability of the computer to distinguish two different solutions. Here, the optimization process is stopped at a success level for which is arbitrarily considered that the numerical solution y and the dataset $\hat{y}$ are no longer discernible. The method is stopped when the norm of the gradient of the cost function is small enough. Figure 9 shows the optimal behavior in three cases for which the process is stopped for a threshold value ϵ of $\|\nabla J(\Psi)\|$. With rather close but different values of the viscosity coefficient taken as control, indiscernible solutions of the model on the first decimals are obtained. It is shown that the optimization process is efficient if the required control has a real influence. In this case, the coefficient is more quickly optimized where the diffusion process is predominant.

To end this reflection concerning the viscosity coefficient influence, the previous problem, with v as a constant, is looked at again. The dataset is computed with $v = 5\ \text{m}^2\ \text{s}^{-1}$. The maximal value of the Reynolds number is around 2. Simulations are performed with different values of the viscosity. The value of the cost function J is computed for each simulation. Figure 10 shows that for a large range of v values (from 1 to 15 $\text{m}^2\ \text{s}^{-1}$), the cost function is small enough ($J \leq 0.1$) to consider that a rather good simulation is obtained. So, it is not necessary to accurately optimize this control. A range of values is indeed sufficient.

3.3. Simultaneous optimization by the three controls

The previous sensibility study shows a higher difficulty in optimizing the viscosity coefficient than the boundary and initial conditions. An optimal control, simultaneously using the three types of controls, is presented.

First, the numerical simulation of the dataset with the 'true' controls is realized

$$y(x, 0) = 1 - \frac{2x}{L} \quad \text{for } x \in (0, L),$$

$$y(0, t) = 1 - \frac{2t}{T} \quad \text{for } t \in (0, T),$$

$$y(L, t) = \frac{2t}{T} - 1 \quad \text{for } t \in (0, T),$$

$$v(x) = 1 + 5\frac{x}{L} \quad \text{for } x \in (0, L).$$

In order to perform a numerical improvement and to be clearer on the following purpose, v is only space dependent. The problem is always the same. It consists of finding the 'true' controls again, in order to minimize the cost function. This experiment will show that the optimal simulation can be obtained by optimal controls different from the 'true' controls. The simultaneous optimization by the three controls leads to the following results:

- The dataset is found by the optimal simulation.
- Initial conditions control are found.
- Boundary conditions controls are not found.
- The viscosity coefficient is not found with accuracy, especially near the boundaries (see Figure 11).

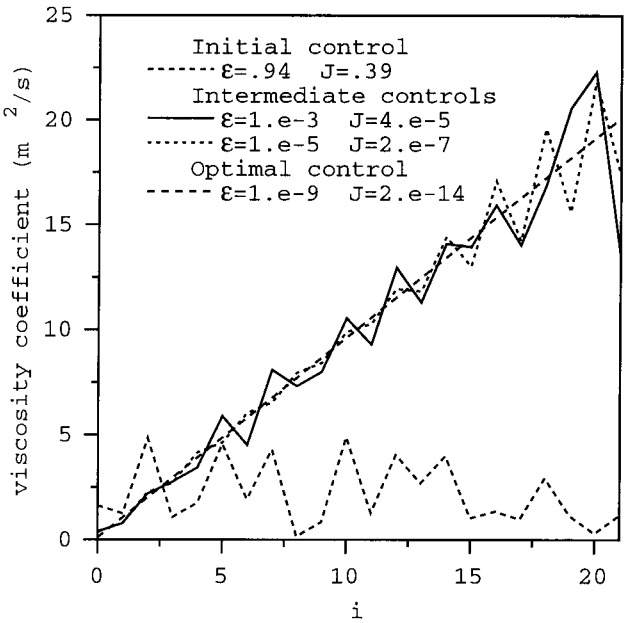

Figure 9. Optimal viscosity coefficient according to the required accuracy.

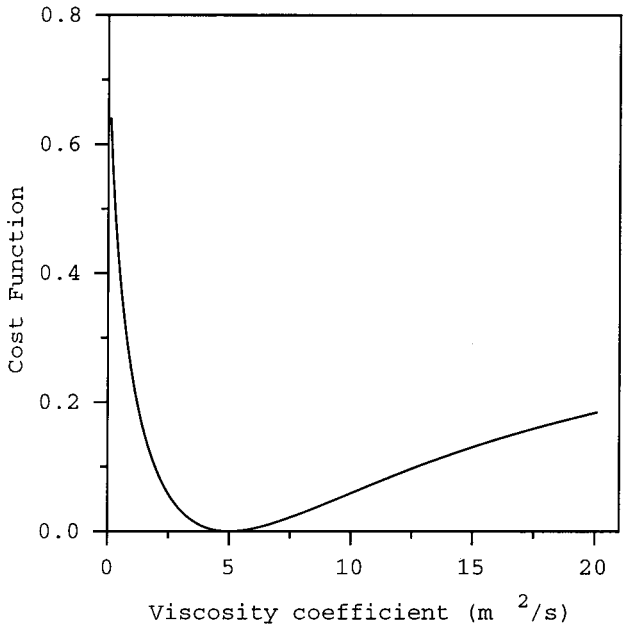

Figure 10. Cost function depending on the constant viscosity coefficient.

A part of the optimal control aim is realized. The dataset is efficiently simulated but with optimal controls different from the ones expected. It is demonstrated here that there are some discrete problems for which optimal controls uniqueness is not ensured.

For example, the discretized equation on the right boundary (for $i = I$) described in Section 2.2 is

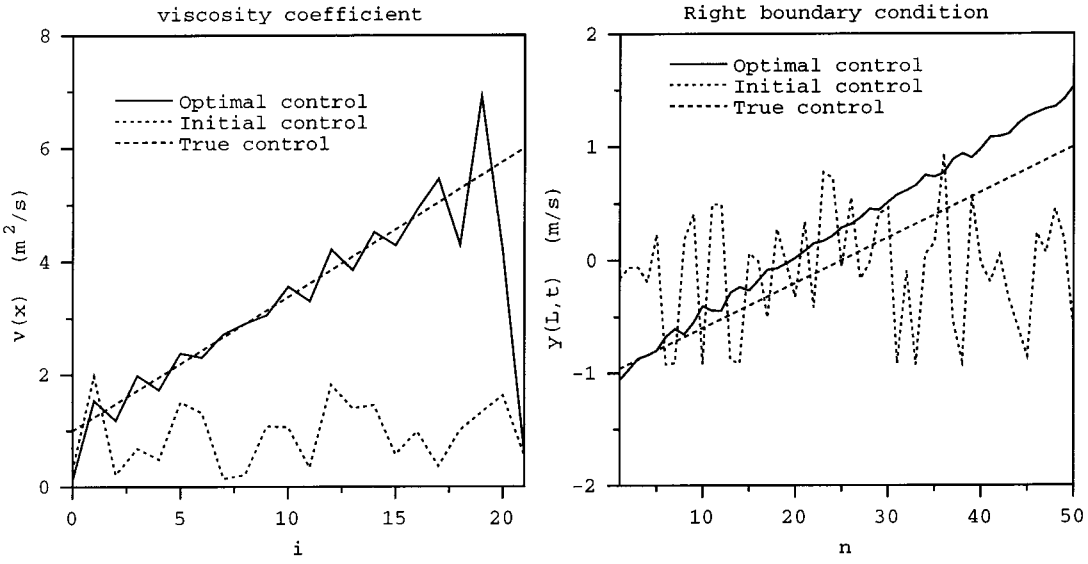

Figure 11. Viscosity coefficient and right boundary conditions.

$$\frac{1}{\Delta t}(y_I^n - y_I^{n-1}) - \frac{1}{2\Delta x^2}((v_I + v_{I+1})(y_{I+1}^n - y_I^n) - (v_{I-1} + v_I)(y_I^n - y_{I-1}^n))$$

$$+ \frac{1}{2\Delta x}(y_I^n y_I^n - y_{I-1}^n y_{I-1}^n) = 0.$$

This equation is controlled by the boundary condition y_{I+1}^n and the three viscosity coefficients v_{I-1}, v_I, and v_{I+1}. The three of them are in fact combined together in the control term $(v_I + v_{I+1})y_{I+1}^n$. So, as soon as the set $(v_I, v_{I+1}, y_{I+1}^n)$ verifies

$$(v_I + v_{I+1})y_{I+1}^n = (v_{I_{\text{true}}} + v_{I+1_{\text{true}}})y_{I+1_{\text{true}}}^n,$$

it is considered by the algorithm as an optimal set for control. This means that there are many sets of controls leading to the same direct model solution. The control by boundary conditions and the control by the viscosity coefficient values on lateral meshes are redundant. To solve this problem, controls on the discretized equations model have to be pointed out. To uncouple these controls, controls set have to be chosen.

Here, for example, the boundary conditions can be kept as controls. So, there is no search to optimize the viscosity coefficient values on the lateral meshes as they are extrapolated from those taken inside the domain. The boundary conditions values and the viscosity coefficient can be chosen arbitrarily as controls on all the domain. In this case, the optimal viscosity coefficient allows the appropriate simulation to be found, even if the boundary conditions are wrong.

4. PERSPECTIVES AND CONCLUSIONS

The optimal control technique applied on a rather simple equation gives useful information in order to deal with more complex models with *in-situ* collected data assimilation.

The discretization process is very delicate and must respect several rules

- The discretization scheme is chosen as implicit on the first time step and decentered for the advection term on the boundaries.
- The controls do not always have a real influence on all the domains of time and space. Here, where the diffusion process is not preponderant, it is needless to search for the optimal values of the viscosity coefficient. This conclusion is transposable to other situations of the models controllability. If a long simulation that focuses on the ultimate state of the system is considered, it could be meaningless to search for the initial conditions.
- The simultaneous process of optimal control by different types of parameters can be altered by a deficient discretization that does not ensure the uniqueness of the controls. The discretized model can be no longer injective.

So, it is necessary to identify the controls and to verify that they have a real and direct influence on the model, e.g. by computing the directional derivatives of the solution along the controls. In the present case, the priority is to fit the initial and boundary conditions and then the functional parameter. In other studies, e.g. Das and Lardner [15], the controlled model is more sensitive to the functional parameters. After that, it is necessary to point out possible influences or redundancies between the chosen controls.

In most hydrodynamical models, principally in 3D oceanic circulation models, the viscosity coefficients' values are linked to the physical and turbulent state of the system and computed using a turbulence model. Mellor and Yamada [16] or Nihoul *et al.* [17] give examples of such a computation of v by a turbulence model.

In the present paper, the data assimilation technique of optimal control applied to this viscosity coefficient gives results that are more than satisfying. So one can wonder whether the optimization of the viscosity coefficient by a data assimilation method can replace a classical model of turbulence.

Indeed, it is demonstrated that this variational method is applicable to theoretical experiments. However, we are pessimistic of a more concrete application, particularly for 3D oceanic circulation models. For a case of a time and space dependence, the control vector size becomes very large and a reasonable optimization can only be obtained if the data have quality and quantity. Furthermore, turbulence models are based on more physical considerations and reduce the free degrees number of the control. So, it is more realistic to use a turbulence model to determine a viscosity coefficient with a physical sense. All the same, these turbulence models can be improved using optimal control to fit the modeling constants appearing in some empirical formulations. Finally, it seems that optimal control cannot replace the turbulence modeling but can improve it.

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